Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021–8928(02)00010-2

J. Appl. Maths Mechs, Vol. 66, No. 1, pp. 71–81, 2002

0021–8928/02/\$—see front matter

THE ACTION OF PERIODIC SURFACE PRESSURES ON A FLOATING ELASTIC PLATFORM†

I. V. STUROVA

Novosibirsk

e-mail: sturova@hydro.nsc.ru

(*Received* 30 *January* 2001)

A solution of the linear hydroelastic problem of the steady forced oscillations of a floating platform acted upon by a localized external load is given. The platform is assumed to be fairly thin and is modelled by an elastic plate with free edges. The method employed involves decomposing the region occupied by the liquid into subregions, bounded above either by a free surface or an elastic plate. A solution is obtained using an expansion of the required velocity potentials into eigenfunctions of the corresponding boundary-value problems. A beam plate of finite and semi-infinite length is considered in the plane case and a circular plate in the three-dimensional case. The solutions obtained for shallow water and for a liquid of finite depth are compared.

The action of dynamic loads on a thin elastic floating plate has been investigated in detail as it applies to an ice sheet (see, for example, [1–3]). It was assumed that the ice completely covers the free surface of the liquid. In connection with the construction of extremely large artificial floating structures, such as floating islands, the need arises to investigate the influence of unsteady dynamic loads on an elastic plate of finite dimensions. Despite the fairly complete solution in the linear formulation of the hydroelastic problem of the action of unsteady loads on an unbounded plate, there are no solutions for finite regions.

In this paper we obtain a solution of the linear hydroelastic problem of the steady oscillations of a bounded floating plate acted upon by a periodic external load.

1. FORMULATION OF THE PROBLEM

Suppose a thin elastic plate, occupying a region Ω_1 and bounded by a contour Γ floats on the surface of a layer of an ideal incompressible liquid of depth h. The region Ω_2 outside the plate is the free surface of the liquid. We will assume that periodic normal stresses, with frequency ω, having the form

$$p(x, y, t) = P(x, y)\exp(i\omega t) \tag{1.1}$$

act on the plate, where x and y are horizontal coordinates and t is the time. We will investigate the oscillations of the liquid and the plate, due to these stresses, assuming that the motion that occurs is steady. The motion of the liquid is assumed to be potential, while the velocity of the liquid particles and the bending of the plate are assumed to be small.

The velocity potentials $\phi_j(\mathbf{x}, t)$, which describe the motion of the liquid under the plate ($j = 1$) and in the region bounded by the free surface ($j = 2$), will be sought in the form $\phi_j(\mathbf{x}, t) = \Phi_j(\mathbf{x}) \exp(i\omega t)$, where $\mathbf{x} = (x, y, z)$, the z axis is directed vertically upwards and the origin of coordinates is at the bottom of the basin. The normal bending of the plate $w(x, y, t) = W(x, y) \exp(i\omega t)$ and the elevation of the free surface $\eta(x, y, t) = \zeta(x, y) \exp(i\omega t)$ are found from the relations

$$W = -\frac{i}{\omega}\frac{\partial\Phi_1}{\partial z}\bigg|_{z=h}, \quad \zeta = -\frac{i}{\omega}\frac{\partial\Phi_2}{\partial z}\bigg|_{z=h} \tag{1.2}$$

By linear wave theory, to determine $\Phi_j(\mathbf{x})$ ($j = 1, 2$) we need to solve the system of equations

$$\Delta\Phi_j + \frac{\partial^2\Phi_j}{\partial z^2} = 0 \quad \left(x, y \in \Omega_j,\ 0 < z < h\right), \quad \Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{1.3}$$

†*Prikl. Mat. Mekh.* Vol. 66, No. 1, pp. 75–86, 2002.

with boundary conditions

$$D\Delta^2 W - \rho_1 h_1 \omega^2 W + \rho g W + i\rho\omega\Phi_1 = -P(x,y) \quad (x, y \in \Omega_1, \; z = h) \tag{1.4}$$

$$\frac{\partial \Phi_2}{\partial z} - \frac{\omega^2}{g}\Phi_2 = 0 \quad (x, y \in \Omega_2, \; z = h) \tag{1.5}$$

$$\frac{\partial \Phi_j}{\partial z} = 0 \quad (z = 0) \tag{1.6}$$

where $D = E h_1^3/[12(1 - \nu^2)]$, E, ρ_1, h_1, ν is the modulus of normal elasticity, the density, the thickness and Poisson's ratio of the plate, ρ is the density of the liquid and g is the acceleration due to gravity. The following matching conditions, which denote the continuity of the pressure and the horizontal velocity along the normal n in the contour Γ, must be satisfied on the side surface of the vertical column of liquid with section Ω_1

$$\Phi_1 = \Phi_2, \quad \frac{\partial \Phi_1}{\partial n} = \frac{\partial \Phi_2}{\partial n} \quad (x, y \in \Gamma, \; 0 < z < h) \tag{1.7}$$

On the edges of the plate we have the free-edge conditions, i.e. the bending moment and the shearing force vanish (see, for example, [4])

$$\Delta W = \nu_1 \left[\frac{\partial^2 W}{\partial s^2} + \alpha'(s)\frac{\partial W}{\partial n}\right], \quad \frac{\partial \Delta W}{\partial n} = \nu_1 \frac{\partial}{\partial s}\left[\alpha'(s)\frac{\partial W}{\partial s} - \frac{\partial^2 W}{\partial s \partial n}\right] \quad (x, y \in \Gamma) \tag{1.8}$$

where $\alpha(s)$ is the angle of inclination of the outward normal to the x axis, s is the arc coordinate of the contour Γ, $\nu_1 = 1 - \nu$, and the prime denotes differentiation with respect to s. The following radiation condition must be satisfied far from the plate

$$\lim_{r \to \infty} \sqrt{r}\left(\frac{\partial}{\partial r} - ik_0\right)\Phi_2 = 0, \quad r = \sqrt{x^2 + y^2} \tag{1.9}$$

where k_0 is the wave number of the gravitational surface wave, defined as the positive real root of the equation

$$\omega^2 = gk \, \mathrm{th}(kh) \tag{1.10}$$

The radiation condition means that, as $r \to \infty$, the waves are diverging. The power expended in generating these waves is [5]

$$T = \frac{i}{2}\rho\omega r \int_0^{2\pi} d\theta \int_0^h \Phi_2^* \frac{\partial \Phi_2}{\partial r} dz \quad (r \to \infty), \quad \theta = \mathrm{arctg}\frac{y}{x} \tag{1.11}$$

where the asterisk denotes complex conjugation. An alternative method of determining the power in terms of the energy of plane progressive waves is described in [6]. It is well known that the mean energy flux F per oscillation period for a plane wave with amplitude ξ_0 is

$$F = \frac{1}{4}\rho g \omega \zeta_0^2 \left[\frac{1}{k_0} + \frac{2h}{\mathrm{sh}(2k_0 h)}\right]$$

and the power consumed is given by the expression

$$T = r \int_0^{2\pi} F d\theta \tag{1.12}$$

2. THE SHALLOW-WATER APPROXIMATION

If the depth of the liquid is sufficiently small, we can use shallow-water theory, according to which the required velocity potentials are independent of the vertical coordinate z. In this approximation, we can easily take into account the settling of the plate d. To determine $\Phi_j(x,y)$ $(j = 1, 2)$ we obtain the following problem (for more detail see, for example, [4])

$$D\Delta^3\Phi_1 + \left(\rho g - \rho_1 h_1 \omega^2\right)\Delta\Phi_1 + \frac{\rho\omega^2}{h_2}\Phi_1 = \frac{i\omega}{h_2}P(x,y) \quad \left(x, y \in \Omega_1\right)$$

$$\Delta\Phi_2 + \frac{\omega^2}{gh}\Phi_2 = 0 \quad \left(x, y \in \Omega_2\right), \quad h_2 = h - d$$

Matching conditions (1.7) take the form

$$\Phi_1 = \Phi_2, \quad \frac{\partial\Phi_1}{\partial n} = \frac{h}{h_2}\frac{\partial\Phi_2}{\partial n} \quad (x, y \in \Gamma) \tag{2.1}$$

The free-edge conditions (1.8) do not change, but in the radiation condition (1.9) we must put

$$k_0 = \omega / \sqrt{gh} \tag{2.2}$$

The normal sag of the plate is equal to

$$W = \frac{ih_2}{\omega}\Delta\Phi_1$$

3. A BEAM PLATE

We will consider the plane problem in which there is no dependence on the y coordinate. We will assume that the function $P(x)$ is even and non-zero only when $|x| < 1$. The length of the plate is L and its left end has the horizontal coordinate $x = -L_1$, while the right coordinate is $x = L_2 = L - L_1$ (L_1, $L_2 > l$). To solve the plane problem it is convenient to divide the region occupied by the liquid into three parts: S_1 $(-L_1 < x < L_2)$, S_2 $(x < -L_1)$, S_3 $(x > L_2)$; in each of these the velocity potential will be denoted by $\Phi_j(x, z)$ $(j = 1, 2, 3)$. Inside the liquid these functions satisfy the two-dimensional Laplace equation

$$\frac{\partial^2\Phi_j}{\partial x^2} + \frac{\partial^2\Phi_j}{\partial z^2} = 0 \quad \left(x, y \in S_j\right)$$

Using relations (1.2) we will write boundary condition (1.4) in the form

$$\left[D\frac{\partial^4}{\partial x^4} - \rho_1 h_1 \omega^2 + \rho g\right]\frac{\partial\Phi_1}{\partial z} - \rho\omega^2\Phi_1 = -i\omega P(x) \quad \left(-L_1 < x < L_2,\ z = h\right)$$

In regions S_2 and S_3, bounded by the free surface, when $z = h$ condition (1.5) is satisfied for Φ_2 and Φ_3, and on the bottom condition (1.6) is satisfied for all functions Φ_j. The matching conditions (1.7) now have the form

$$\Phi_1 = \Phi_2, \quad \frac{\partial\Phi_1}{\partial x} = \frac{\partial\Phi_2}{\partial x} \quad \left(x = -L_1,\ 0 \leqslant z \leqslant h\right) \tag{3.1}$$

$$\Phi_1 = \Phi_3, \quad \frac{\partial\Phi_1}{\partial x} = \frac{\partial\Phi_3}{\partial x} \quad \left(x = L_2,\ 0 \leqslant z \leqslant h\right) \tag{3.2}$$

The free-edge conditions (1.8) are considerably simplified and take the form

$$\frac{\partial^3\Phi_1}{\partial x^2\partial z}=\frac{\partial^4\Phi_1}{\partial x^3\partial z}=0 \quad (x=-L_1,\ x=L_2,\ z=h)$$

The functions Φ_j (j = 1, 2, 3), as previously [7, 8], will be sought in the form of an expansion in eigenfunctions of the corresponding boundary-value problems. In the region of the liquid S_1, bounded by the elastic plate, the normalized natural vertical modes have the form

$$q_m(z)=\mathrm{ch}(\mu_m z)/\sqrt{\Lambda(\mu_m)},\quad \Lambda(\mu)=h/2+\mathrm{sh}(2\mu h)/(4\mu)$$

where the eigenvalues μ_m are the roots of the equation

$$\omega^2=S(\mu);\quad S(\mu)=\left(D\mu^4+\rho g\right)\mu\,\mathrm{th}(\mu h)/Q(\mu),\quad Q(\mu)=\rho+\rho_1 h_1\mu\,\mathrm{th}(\mu h)$$

The quantities μ_{-1} and $\mu_{-2}=-\mu_{-1}^*$ are complex and belong to the first and second quadrant, respectively, of the complex plane μ, the real positive root μ_0 is the wave number of the flexural-gravitational wave, while the quantities μ_1, μ_2, ... are pure imaginary roots.

We will write the general solution for Φ_1 in the form

$$\Phi_1(x,z)=\sum_{m=-2}^{\infty}\left[A_m\exp(-i\mu_m x)+B_m\exp(i\mu_m x)\right]q_m(z)+\Phi_0(x,z) \tag{3.3}$$

where $\Phi_0(x, z)$ is the solution of the problem of the action of periodic pressures on an unbounded beam plate. The solution of this problem is well known [1, 2, 9] and has the form

$$\Phi_0=\omega\left[\frac{i}{\pi}\,\mathrm{p.v.}\int_0^{\infty}\frac{\tilde{P}(\xi)\mathrm{ch}(\xi z)\cos(\xi x)d\xi}{\mathrm{ch}(\xi h)Q(\xi)\left[\omega^2-S(\xi)\right]}-\frac{\tilde{P}(\mu_0)\mathrm{ch}(\mu_0 z)\cos(\mu_0 x)}{\mathrm{ch}(\mu_0 h)Q(\mu_0)S'(\mu_0)}\right] \tag{3.4}$$

$$\tilde{P}(\xi)=2\int_0^{l}P(x)\cos(\xi x)dx,\quad S'(\mu_0)\equiv\left.\frac{dS}{d\xi}\right|_{\xi=\mu_0}$$

($\tilde{P}(\xi)$ is the Fourier transform of the function $P(x)$, and p.v. denotes the integral in the sense of the principal value). The second term in the square brackets in (3.4) is necessary in order to satisfy the radiation condition, since the integrand always has a simple pole at the point $\xi=\mu_0$. In the far field of an infinite plate a single system of flexural-gravitational waves with wavelength $2\pi/\mu_0$ is generated, and the normal sag of the plate is described by the expression

$$W(x)=\frac{i\mu_0\tilde{P}(\mu_0)\mathrm{th}(\mu_0 h)}{Q(\mu_0)S'(\mu_0)}\exp(\mp i\mu_0 x)\quad (x\to\pm\infty)$$

Examples of the calculations of the amplitudes of these waves were given previously in [9].

The general solutions for Φ_2 and Φ_3 in regions bounded by the free surface of the liquid can be represented in the form

$$\Phi_2(x,z)=C_0\exp(ik_0 x)f_0(z)+\sum_{m=1}^{\infty}C_m\exp(-ik_m x)f_m(z) \tag{3.5}$$

$$\Phi_3(x,z)=F_0\exp(-ik_0 x)f_0(z)+\sum_{m=1}^{\infty}F_m\exp(ik_m x)f_m(z) \tag{3.6}$$

where $f_m=\mathrm{ch}(k_m z)/\sqrt{\Lambda(k_m)}$; $k_1, k_2, \dots$ are pure imaginary roots of Eq. (1.10). The wave mades related to k_m and μ_m ($m \geqslant 1$) are called edge waves.

For a numerical solution we used the reduction method and the infinite series in (3.3), (3.5) and (3.6) are replaced by finite sums with a number of terms M. The matching conditions (3.1) and (3.2) are satisfied in the integral sense, i.e. they are multiplied successively by the functions $f_m(z)$, $q_m(z)$ ($0 \leqslant m \leqslant M$) and integrated in the interval $0 \leqslant z \leqslant h$. Finally, the problem is reduced to solving a system of $4(M+2)$ linear equations.

After calculating all the unknown constants in (3.3), (3.5) and (3.6) we can determine the normal sag of the plate and the vertical displacements of the free surface of the liquid, taking (1.2) into account. The amplitudes of the surface waves $\zeta_0^{\pm}$ far from the plate as $x \to \pm\infty$ are given by the expressions

$$\zeta_0^+ = k_0 \operatorname{sh}(k_0 h)|C_0|/\omega, \quad \zeta_0^- = k_0 \operatorname{sh}(k_0 h)|F_0|/\omega$$

A special case of this problem when $L_1 \to -\infty$ is the problem of the action of surface pressures on a semi-infinite beam plate. In this case the region occupied by the liquid is divided into two parts: S_1 ($x < L_2$) and S_3 ($x > L_2$). The representation for Φ_3, as previously, has the form (3.6), while for Φ_1 in (3.3) we must omit terms with coefficients B_{-2}, B_{-1}, A_0 and B_m ($m \geqslant 1$). The matching conditions (3.2) must be satisfied at the boundary of the regions S_1 and S_3. To solve this problem we also used the method of integral splicing, and the problem was reduced to a system of $2(M + 2)$ linear equations.

In the shallow-water approximation there are no edge waves, and for a plate of finite length representations (3.3), (3.5) and (3.6) have the simpler form

$$\Phi_1(x,z) = \sum_{m=-2}^{0} [A_m \exp(-i\mu_m x) + B_m \exp(i\mu_m x)] + \Phi_0(x,z) \tag{3.7}$$

$$\Phi_2 = C_0 \exp(ik_0 x), \quad \Phi_3 = F_0 \exp(-ik_0 x) \tag{3.8}$$

The quantity k_0 is given by expression (2.2), while μ_m ($m = -2, -1, 0$) are the solutions of the cubic equation

$$D\chi^3 + (\rho g - \omega^2 \rho_1 h_1)\chi - \omega^2 \rho / h_2 = 0$$

It was well known [1], that this equation has one positive root χ_1 and two complex-conjugate roots χ_2 and χ_3, in which case $\mu_0 = \sqrt{\chi_1}$, $\mu_{-1} = \sqrt{\chi_2}$, $\mu_{-2} = -\sqrt{\chi_3}$. The solution $\Phi_0(x)$ for an infinite plate has the form [1]

$$\Phi_0 = \frac{\omega}{h_2}\left[\frac{i}{\pi}\,\text{p.v.}\int_0^\infty \frac{\tilde{P}(\xi)\cos(\xi x)d\xi}{Z(\xi)[\omega^2 - \Upsilon(\xi)]} - \frac{\tilde{P}(\mu_0)\cos(\mu_0 x)}{Z(\mu_0)\Upsilon'(\mu_0)}\right]$$

where

$$Z(\xi) = \rho_1 h_1 \xi^2 + \rho / h_2, \quad \Upsilon(\xi) = \xi^2(D\xi^4 + \rho g)/Z(\xi), \quad \Upsilon'(\mu_0) \equiv d\Upsilon / d\xi\,|_{\xi=\mu_0}$$

The unknown coefficients in (3.7) and (3.8) are found from the matching conditions (2.1), which, in the plane case, have the form

$$\Phi_1 = \Phi_2, \quad \frac{\partial \Phi_1}{\partial x} = \frac{h}{h_2}\frac{\partial \Phi_2}{\partial x} \quad (x = -L_1)$$

$$\Phi_1 = \Phi_3, \quad \frac{\partial \Phi_1}{\partial x} = \frac{h}{h_2}\frac{\partial \Phi_3}{\partial x} \quad (x = L_2)$$

and the free-edge conditions

$$\frac{\partial^4 \Phi_1}{\partial x^4} = \frac{\partial^5 \Phi_1}{\partial x^5} = 0 \quad (x = -L_1, \ x = L_2)$$

In the shallow-water approximation the problem reduces to solving a system of eight linear equations for a finite plate and four equations for a semi-infinite plate.

4. A CIRCULAR PLATE

For the three-dimensional problem we will consider the simplest shape, namely, a circular plate.

In addition to a Cartesian system of coordinates x, y, z it is convenient to introduce a cylindrical system r, θ, z. The common vertical axis z passes through the centre of the plate of radius r_0. We will assume that normal stresses are applied to the plate only in a circle of radius l, the centre of which is at the point $x = L_1, y = 0$ $(l + L_1 < r_0)$, while the function $P(x, y)$ in (1.1) depends only on $R = \sqrt{(x-L_1)^2 + y^2}$.

The matching conditions (1.7) for a circular plate take the form

$$\Phi_1 = \Phi_2, \quad \frac{\partial \Phi_1}{\partial r} = \frac{\partial \Phi_2}{\partial r} \quad (r = r_0, \ 0 \leq z \leq h) \tag{4.1}$$

while the free-edge conditions (1.8) take the form

$$\Delta W - \frac{\nu_1}{r_0^2}\frac{\partial^2 W}{\partial \theta^2} - \frac{\nu_1}{r_0}\frac{\partial W}{\partial r} = \frac{\partial \Delta W}{\partial r} - \frac{\nu_1}{r_0^3}\frac{\partial^2 W}{\partial \theta^2} + \frac{\nu_1}{r_0^2}\frac{\partial^3 W}{\partial \theta^2 \partial r} = 0 \quad (r = r_0) \tag{4.2}$$

Using the approach proposed in [7], the solution of Eqs (1.3) will be sought in the form

$$\Phi_1 = \sum_{m=-2}^{\infty} X_m(r,\theta) q_m(z) + \Phi_0(R,z), \quad \Phi_2 = \sum_{m=0}^{\infty} Z_m(r,\theta) f_m(z)$$

where the functions X_m and Y_m satisfy Helmholtz' equations

$$\Delta X_m + \mu_m^2 X_m = 0 \ (m \geq -2), \quad \Delta Z_m + k_m^2 Z_m = 0 \ \ (m \geq 0) \tag{4.3}$$

The function Φ_0 is the solution of the axisymmetric problem of periodic pressures, acting on an infinite plate. It is well known that [1, 2]

$$\Phi_0(R,z) = \frac{\omega}{2}\left[\frac{i}{\pi}\,\mathrm{p.v.}\int_0^{\infty} \frac{\xi \tilde{P}(\xi) J_0(\xi R)\operatorname{ch}(\xi z)d\xi}{\operatorname{ch}(\xi h)Q(\xi)[\omega^2 - S(\xi)]} - \frac{\mu_0 \tilde{P}(\mu_0) J_0(\mu_0 R)\operatorname{ch}(\mu_0 z)}{\operatorname{ch}(\mu_0 h)Q(\mu_0)S'(\mu_0)}\right] \tag{4.4}$$

$$\tilde{P}(\xi) = 2\pi \int_0^{l} RP(R)J_0(\xi R)dR$$

where J_0 is the Bassel function of the first kind of zero order.

We will seek a solution of Eqs (4.3) in the form

$$X_m = \sum_{n=0}^{\infty} A_n^{(m)} J_n(\mu_m r)\cos(n\theta) \ \ (m \geq -2)$$

$$Z_0 = \sum_{n=0}^{\infty} B_n^{(0)} H_n^{(2)}(k_0 r)\cos(n\theta), \quad Z_m = \sum_{n=0}^{\infty} B_n^{(m)} K_n(|k_m| r)\cos(n\theta) \ \ (m \geq 1)$$

where $H_n^{(2)}$ is the Hankel function of the second kind and K_n is the modified Bessel function of the second kind.

We will expand the expression for Φ_0 and (4.4) in a Fourier series

$$\Phi_0(R,z) = \sum_{n=0}^{\infty} D_n(r,z)\cos(n\theta)$$

$$D_n = \omega\varepsilon_n\left[\frac{i}{\pi}\,\mathrm{p.v.}\frac{\xi\tilde{P}(\xi)\operatorname{ch}(\xi z)}{\operatorname{ch}(\xi h)Q(\xi)[\omega^2 - S(\xi)]} J_n(\xi r)J_n(\xi L_1)d\xi - \right.$$

$$\left. - \frac{\mu_0 \tilde{P}(\mu_0)\operatorname{ch}(\mu_0 z)}{\operatorname{ch}(\mu_0 h)Q(\mu_0)S'(\mu_0)} J_n(\mu_0 r)J_n(\mu_0 L_1)\right]$$

$$\varepsilon_0 = 1/2, \quad \varepsilon_n = 1 \ (n \geq 1)$$

To determine the unknown coefficients $A_n^{(m)}$ and $B_n^{(m)}$ we use the matching conditions (4.1), which are satisfied in the same way as described in Section 3 in the integral sense, and the free-edge conditions (4.2). In the systems of equations obtained we must collect the coefficients of like values of $\cos(n\theta)$, and for each angular harmonic, using the reduction method, the problem is reduced to solving a system of $2(M + 2)$ linear equations.

When the pressure spot is situated at the centre of the plate ($L_1 = 0$), the problem becomes axisymmetrical and only the coefficients with $n = 0$ will be non-zero. When $L_1 \neq 0$, N angular harmonics were taken into account in the numerical solution.

Far from the plate, when $r \to \infty$, the vertical displacements of the free surface have the form

$$\zeta = -\frac{ik_0 \operatorname{sh}(k_0 h)}{\omega\sqrt{\Lambda(k_0)}} \sum_{n=0}^{\infty} B_n^{(0)} H_n^{(2)}(k_0 r)\cos(n\theta)$$

Taking into account the asymptotic representation of the Hankel function for large values of the argument, we obtain

$$\zeta \approx -\frac{i}{\omega}\sqrt{\frac{2k_0}{\pi r\Lambda(k_0)}}\operatorname{sh}(k_0 h)\exp\left[i\left(\frac{\pi}{4} - k_0 r\right)\right]H(\theta) \quad (r \to \infty)$$

$$H(\theta) = \sum_{n=0}^{\infty} B_n^{(0)} \exp\frac{in\pi}{2}\cos(n\theta)$$

The amplitude of the surface wave in the far field has the form

$$\zeta_0 \approx \frac{1}{\omega}\sqrt{\frac{2k_0}{\pi r\Lambda(k_0)}}\operatorname{sh}(k_0 h)\,|\,H(\theta)\,|$$

The power consumed in generating the waves, according to expression (1.12), is equal to

$$T = \frac{\rho\omega}{\pi}\int_0^{2\pi} |\,H(\theta)\,|^2\, d\theta = \rho\omega\left[2\,|\,B_0^{(0)}\,|^2 + \sum_{n=1}^{\infty} |\,B_0^{(0)}\,|^2\right]$$

The result is also obtained when using expression (1.11).

In the shallow-water approximation, we must introduce changes similar to those in Section 3, and, as a result, it is necessary to solve a system of four linear equation for each angular harmonic.

5. NUMERICAL RESULTS

We used the following pressure distributions when making the numerical calculations:

in the plane case

$$P(x) = a\left[1 - \left(\frac{x}{l}\right)^2\right] \quad (|\,x\,| < l), \quad \tilde{P}(\xi) = \frac{4a}{l\xi^2}\left[\frac{\sin(\xi l)}{\xi l} - \cos(\xi l)\right]$$

in the three-dimensional case

$$P(R) = a\left[1 - \left(\frac{R}{l}\right)^2\right] \quad (R < l), \quad \tilde{P}(\xi) = 4\pi a\frac{J_2(\xi l)}{\xi^2}$$

where a is a dimensionless factor.

The values of the initial parameters were taken to be the same as in [4], namely,

$$D = 1.093 \times 103 \text{ kg m}^2/\text{s}^2,\ h = 0.25 \text{ m},\ \rho = 103 \text{ kg/m}^3,\ \rho_1 h_1 = 12.5 \text{ kg/m}^2,\ \nu = 0.3$$

The half-width of the pressure region in all the calculations is equal to $l = 2h$. The settling of the plate was ignored in the shallow-water solution.

Figure 1 shows the amplitudes of the surface waves in the far field $\bar{\zeta}_0^{\pm} = \rho g \zeta_0^{\pm}/a$ as a function of the dimensionless frequency $\bar{\omega} = \sqrt{h/g}$ for a beam plate with $L_1 = 15h$, $L_2 = 5h$ (a, b) and $L_1 + L_2 = 10h$ (c). Curves 1 and 2 represent the solutions for a finite plate, while curves 3 and 4 in Fig. 1(a, c) represent the solutions for a semi-infinite plate. In Figs 1 and 2 the odd numbers on the

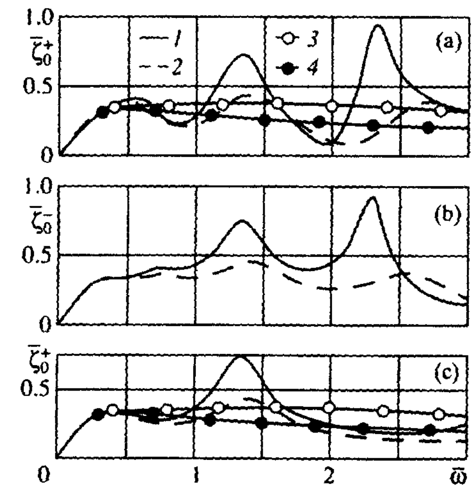

Fig. 1

curves correspond to solutions for a liquid of finite depth taken $M = 20$ edge modes into account, while the even numbers correspond to shallow water.

It wan be seen that in the case of a finite plate the dependence of the amplitudes of the surface waves on the frequency is extremely non-monotonic. This is not observed for a semi-infinite plate.

The amplitudes of the normal sags of the plate $\bar{W} = \rho g W/a$ are represented in Fig. 2(a–c) for three values of the dimensionless frequency $\omega = 1$, 1.3, and 1.6 respectively, in the case of a finite plate with $L_1 = 15h$ and $L_2 = 5h$ (curves 1 and 2), for a semi-infinite plate with $L_2 = 5h$ (curves 3 and 4) and for an unbounded plate (curves 5 and 6). The hatched strip in Figs 2 and 4 shows the region in which the

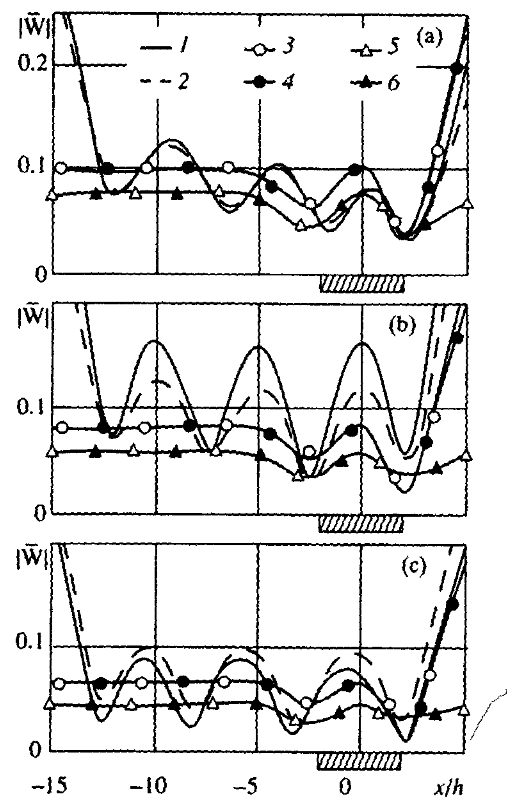

Fig. 2

pressure is applied. It can be seen that the maximum sags occur at the edges of a finite plate and of the three cases shown the greatest sags occur when $\bar{\omega} = 1.3$. This frequency corresponds approximately to the local maximum of the amplitude of the vertical displacements in the far field (Fig. 1a, b). The action of a periodic load leads to much greater sags of the plate when it is of limited dimensions compared with the case of an infinite plate. In the case of a semi-infinite plate the parameters of the oscillations of the free edge almost repeat the values for a finite plate, but the normal sags decrease considerably as one moves to the left from the pressure region and become close to the values in the case of an infinite plate. A similar situation occurs in the diffraction problem when investigating the scattering of surface waves on a floating elastic strip (see, for example, [8]).

It is interesting to note that the solutions for a liquid of finite depth and shallow water differ much less in the behaviour of the plate than of the free surface. This can be explained by the fact that, for gravitational waves, the relative disagreement between the wave numbers of these two solutions reaches 10% when $\bar{\omega} \approx 0.8$, whereas for flexural–gravitational waves, with the parameters used, the same disagreement only occurs when $\bar{\omega} \approx 3.8$.

In Fig. 3(a–d) we show the power $\bar{T} = T\rho\sqrt{g/h^3}\,/\,a^2$ as a function of frequency for a circular plate with $r_0 = 10h$ for $L_1/h = 0, 3, 5$ and 7, respectively. Curves 1 and 2 represent the case of a liquid of finite depth with $M = 10$ and 20, while curves 3 represent the shallow-water approximation. In all the calculations $N = 15$ when $L_1 \neq 0$. Any further increase in N does not change the result. As in the plane problem (compare with Fig. 1), at certain frequencies there is a sharp increase in the wave motions. A change in the number of edge modes has only a small effect in the region of the power peaks.

It is well known [10] that for a circular elastic plate, oscillating in a vacuum, there is a discrete set of real natural frequencies for each angular harmonic. The least natural frequency when $n = 0$ for a plate with free edges with the chosen parameters corresponds to $\bar{\omega} = 2.17$. For an elastic plate, floating on the surface of a heavy liquid, only complex natural frequencies with a positive imaginary part due to

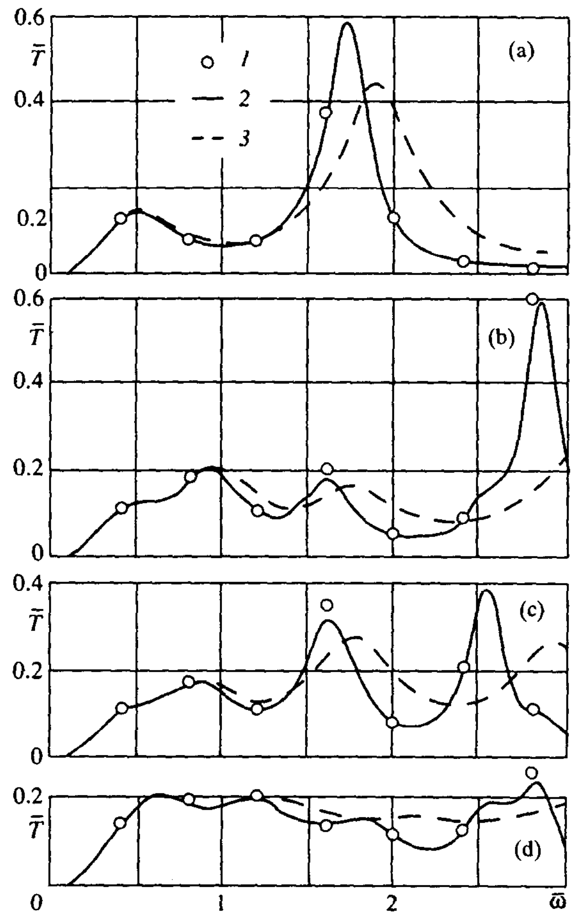

Fig. 3

scattering of the energy by surface waves can exist. Local maxima in the frequency-dependence of the power T may be a reference point in searches for the eigenvalues with minimum imaginary party. Note that the values of the local power maxima decrease the further the pressure spot is from the centre of the plate.

In Fig. 4(a–c) we show the amplitudes of the normal sags of a circular plate along the diameter line $y = 0$ with $r_0 = 10h$ and $L_1 = 5h$ for $\bar{\omega} = 1, 1.6$ and 2.2 respectively. Curves 1 and 2 represent the solutions for a circular plate, and curves 3 and 4 are for an unbounded plate. The odd numbers on the curves correspond to the solutions for a liquid of finite depth, taking into account $M = 20$ edge modes, while the odd numbers correspond to shallow water; in both cases $N = 15$. Like the plane case (compare with Fig. 2) the greatest sags of the plate are observed at its edges. At a frequency $\bar{\omega} = 1.6$, which corresponds approximately to the local power maximum (see Fig. 3c), the sags of the plate are greater than for the other frequencies considered.

In Fig. 5(a–c) we show isolines of the amplitudes of the normal sags of the plate $\bar{W}(r < r_0)$ and the vertical displacements of the free surface $\bar{\zeta}$ $(r > r_0)$ for $\bar{\omega} = 0.5$ and $r_0 - L_1 = 5h$ for three values of the plate radii $r_0/h = 10$, 20 and 30, respectively. The upper halves of each of the figure show the solution for a liquid of finite depth ($M = 10$), while the lower half shows the solution for shallow water; in both solutions $N = 20$. The isolines are drawn in steps of 0.02. The dashed circle corresponds to the region where the pressure is applied. The boundary of the plate is shown by the thicker line. The behaviour

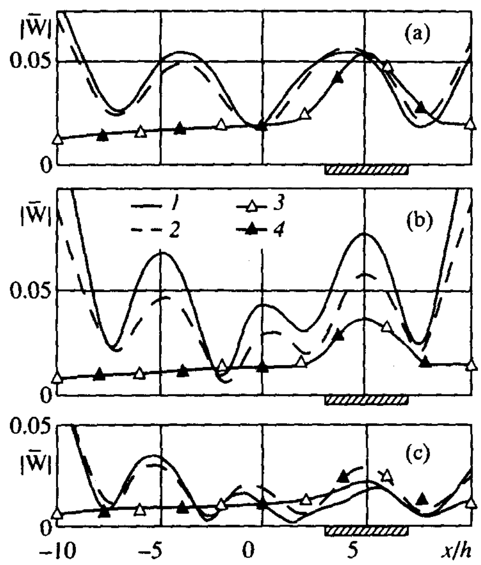

Fig. 4

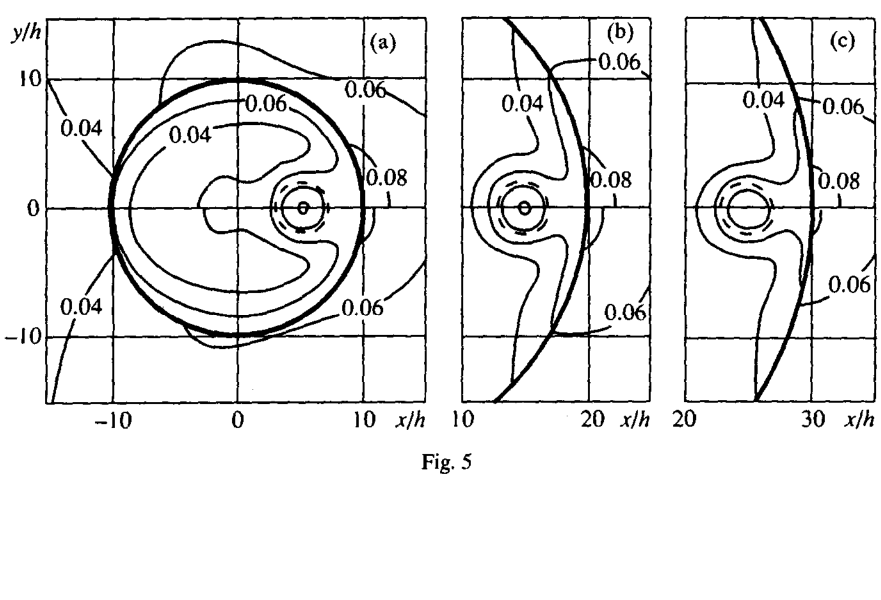

Fig. 5

of the plate is practically same for the two solutions, but small disagreements are observed in the behaviour of the free surface.

For a fixed distance of the pressure spot from the right edge of the plate, the solution of the problem, as the radius of the plate increases, approaches the solution of the problem of the action of pressure on a semi-infinite plate in the region of its straight edge.

6. CONCLUSION

The results obtained show that the limited dimensions of an elastic plate considerably influence the characteristics of its oscillation when acted upon by an external periodic load. The qualitative behaviour of the sags of the plate in the plane and the three-dimensional cases is the same. The finite dimensions of the plate in both cases lead to considerably greater oscillations of the plate (particularly at the edges) compared with an unbounded plate. In some cases the amplitudes of the normal sags of the plate at its edges considerably exceed the corresponding values in the inner part, which indicates possible localization of the oscillations of the plate in the neighbourhood of the edges. For a finite plate there are resonance frequencies of the external load for which the amplitudes of both the normal sags of the plate and the vertical displacements of the free surface of the liquid increase sharply. This phenomenon is more pronounced for a liquid of finite depth than for shallow water.

The proposed approach can be extended to investigate the hydroelastic behaviour of plates of different shape when acted upon by periodic pressures. The oscillations of a rectangular plate, floating on the surface of a liquid of finite depth, can be determined by using the results obtained previously [7]. In the shallow-water approximation, one can consider a plate of arbitrary shape using an approach similar to that described earlier in [4].

This research was supported financially by the Integration Project No. 1 of the Siberian Section of the Russian Academy of Sciences and the "Leading Scientific Schools" Program of the Russian Foundation for Basic Research (00-15-96162).

REFERENCES

1. KHEISIN, D. Ye., *The Dynamics of an Ice Sheet*. Gidrometeoizdat, Leningrad, 1967.
2. CHERKESOV, L. V., *Surface and Internal Waves*. Naukova Dumka, Kiev, 1973.
3. SQUIRE, V. A., HOSKING, R. J., KERR, A. D. et al., *Moving Loads on Ice Plates*. Kluwer, Dordrecht, 1996.
4. STUROVA, I. V., The diffraction of surface waves by an elastic platform floating on shallow water. *Prikl. Mat. Mekh.*, 2001, **65**, 1, 114–122.
5. WEHAUSEN, J. V. and LAITONE, E. V., Surface waves. *Handbuch der Physik*, Vol. 9. Springer, Berlin, 1960, 446–778.
6. KOCHIN, N. Ye., The theory of waves induced by the oscillations of a body under the free surface of a heavy incompressible liquid. In *Collected Papers*, Vol. 2. Izd. Akad. Nauk SSSR, Moscow and Leningrad, 1949, 277–304.
7. KIM, J. W., and ERTEKIN, R. C., An eigenfunction-expansion method for predicting hydroelastic behaviour of a shallow-draft VLFS. *Hydroelasticity in Marine Technology: Proc. 2nd Int. Conf.* Res. Inst. Appl. Mech. Kyushu Univ., Fukuoka, Japan, 1998, 47–59.
8. STUROVA, I. V., Oblique incidence of surface waves on an elastic strip. *Zh. Prikl. Mekh. Tekh. Fiz.*, 1999, **40**, 4, 62–68.
9. BUKATOV, A. Ye., The effect of longitudinal compression on the steady oscillations of an elastic plate floating on the surface of a liquid. *Prikl. Mekh.*, 1981, **17**, 1, 93–98.
10. GONTKEVICH, V. S., *The Natural Oscillations of Plates and Shells. A Handbook*. Naukova, Dumka, Kiev 1964.

Translated by R.C.G.